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SOLVING HEAT TRANSFER PROBLEM IN ULTRASONIC WELDING BASED ON HYBRID SPLINE DIFFERENCE METHOD

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Abstract:

A Hybrid Spline Difference Method is developed to solve a nonlinear equation of welding problem in ultrasonic welding. It is shown that the method has a computational procedure as simply as the finite difference method. In addition, the proposed method can simplify complexity of the traditional spline method calculation, and increase accuracy of the first and second derivatives of space from $O(\Delta x^2)$ of finite difference method to $O(\Delta x^4)$. According to the calculated temperature distribution in the work pieces, during the ultrasonic welding process, the proposed method illustrated that not only its precision is greatly enhanced, but also its concept is very similar to that of the finite difference method. Based on analysis results, it was concluded that the simple and high-accuracy hybrid spline difference method has a strong potential to substitute the traditional finite difference method.

Keywords: *ultrasonic metal welding, hybrid spline difference method, finite difference method.*

1. Introduction

Ultrasonic welding (USW) is a process for joining similar and dissimilar material samples in various industries. Heat generated at welding interface during the welding process is due to plastic deformation and friction from a motion between two contacting work pieces' surface [1]. The heat generation plays an important role in welding process, generally, because it significantly influences on temperature distribution in the weld parts.

Several researchers have studied temperature distributions at the interface and in the work pieces as well as in the horn (sonotrode). The temperature was predicted by using ANSYS finite element models [2-4]. Thermal conductivity and specific heat in these researches were considered as constant, it means the governing heat transfer equations were established are only linear equation.

Actually, many numerical methods have been proposed for solving heat transfer problems. The Finite Difference Method (FDM) has been used to solve heat transfer of complex geometric shapes [5, 6]. In order to increase the accuracy of numerical method, the hybrid differential transformation method, - Taylor transformation method [7, 8], the boundary element method [9] as well as the finite volume method [10] can be effectively used. In addition, almost previous analyses of heat and mass transfer used the FDM because of its simple concept and easy operation; however, the FDM's solutions rarely have high accuracy. On the contrary, with characteristics of smoothness and continuity, the spline method has higher numerical precision than that of FDM; thus, numerical solution of spline [11-14] has been widely applied. However, the spline method has a complicated calculation procedure and an unsolved problem of determination of the optimal parameters. Therefore, in recent years Wang et al. [15-18] constructed a simple procedure of solving the spline difference in a discretization approach similar to finite difference.

This study develops the skill of hybrid spline that makes the first order and second order numerical differential accuracies reach $O(\Delta x)^4$ at the same time. The nonlinear equation of welding problem in ultrasonic welding is analyzed to validate a simple and high-accuracy characteristic of proposed hybrid spline difference method (HSDM).

2. Mathematical Structure

2.1. Construction of Hybrid Spline Difference Method

2.1.1. Original parametric spline

In numerical methods, a single polynomial is usually used to approximate an arbitrary function, and it is found that this approach was sometime unsatisfactory. To overcome this deficiency, the functional region can be divided into many sub-regions which are presented by polynomials and simple functions. Wang and Kahawita [11] hypothesized a traditional cubic spline function is as a simple cubic polynomial such that its curvature after second differential is a linear relationship as

$$\frac{\phi''_i(x) - \phi''_{i-1}}{x - x_{i-1}} = \frac{\phi''_i - \phi''_i(x)}{x_i - x} \quad (1)$$

where $\phi_{i-1}(x)$ and $\phi_i(x)$ are the cubic spline approximation curves in sub-interval $[x_{i-2}, x_{i-1}]$ and $[x_{i-1}, x_i]$, respectively. $\phi''_{i-1}(x)$ and $\phi''_i(x)$ are the second order derivatives of $\phi_{i-1}(x)$ and $\phi_i(x)$, respectively. In recent years, some authors [13, 14, 19] have tried adding a random undetermined parameter, τ, in the traditional spline for raising the accuracy, and assumed the quadratic differential relation as

$$\begin{aligned} &\phi''_i(x,\gamma) + \tau\phi_i(x,\gamma) = \\ &[\phi''_i(x_{i-1},\gamma) + \tau\phi_i(x_{i-1},\gamma)]\left(\frac{x_i - x}{\Delta x_i}\right) + \\ &+[\phi''_i(x_i,\gamma) + \tau\phi_i(x_i,\gamma)]\left(\frac{x - x_{i-1}}{\Delta x_i}\right) \end{aligned} \quad (2)$$

where $\gamma \geq 0$ is a free parameter, x_i is the discrete grid points in the computational domain $[x_0, x_N]$, the interval Δx_i is defined by $x_i - x_{i-1}$, $\phi_i(x,\gamma)$ is unknown function on $[x_{-1}, x_{N+1}]$. Solving Eq. (2) and using the end point relationships $\phi_i(x_{i-1}) = \phi_{i-1}$ and $\phi_i(x_i) = \phi_i$ to determine constants of integration, it can be given:

$$\phi_i(x,\gamma) = z\phi_i + \bar{z}\phi_{i-1} + \Delta x_i^2[g(z)\phi''_i + g(\bar{z})\phi''_{i-1}]/\omega^2 \quad (3)$$

in which $\omega = \Delta x\sqrt{\gamma}$, $z = (x - x_i)/\Delta x$, $\bar{z} = 1 - z$, $g(z) = z - \sin(\omega z)/\sin\omega$. Similarly, we obtain the relation $\phi_{i+1}(x,\gamma)$ with $i + 1$ replacing i in Eq. (3). Then the following fundamental relations of the parameter spline function can be deduced:

$$\phi'_i = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} - \frac{\alpha\Delta x(\phi''_{i+1} - \phi''_{i-1})}{2} \quad (4a)$$

$$\alpha\phi'_{i-1} + 2\beta\phi'_i + \alpha\phi'_{i+1} = \frac{\alpha + \beta}{\Delta x}(\phi_{i+1} - \phi_{i-1}) \quad (4b)$$

$$\alpha\phi''_{i-1} + 2\beta\phi''_i + \alpha\phi''_{i+1} = \frac{1}{\Delta x^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) \quad (4c)$$

where $\alpha = [\omega\csc(\omega) - 1]/\omega^2$, $\beta = [1 - \omega\cot(\omega)]/\omega^2$.

2.1.2. Basic conception of spline difference

In the previous spline method, Eqs. (4) are mainly solved using Eq. (4) combined with the differential equation itself. This procedure is more complicated and it is quite different from the conventional FDM. Thus, in this study, an approximate function of the differential equation is adopted to construct multiple different parametric splines, $\phi(x - x_i/\Delta x, \gamma)$, expressed as

$$\phi(x,\gamma) = \sum_{i=-1}^{N+1} p_i\phi\left(\frac{x - x_i}{\Delta x}, \gamma\right) \quad (5)$$

where p_i, the unknown coefficient, is the spline size value at the grid point i. Substituting Eq. (5) into relational expression (4), we can obtain

$$\begin{aligned} &\phi_i = \alpha p_{i-1} + 2\beta p_i + \alpha p_{i+1}; \\ &\phi'_i = \frac{p_{i+1} - p_{i-1}}{2\Delta x}; \phi''_i = \frac{p_{i-1} - 2p_i + p_{i+1}}{\Delta x^2} \end{aligned} \quad (6)$$

Eq. (6) is the discrete relationship of the spline at the grid point. The first and second differential discrete forms of the function are closed to the traditional FDM.

2.1.3. Concept of hybrid spline difference

Assuming $\phi(x)$ is the exact solution of the differential equation, according to the Taylor series expression at x_i, the truncation error of the first and the second derivatives of the approximate function can be given as

$$\begin{aligned} &\phi'_i - \phi'(x_i) = (\varepsilon - 1)\phi'(x_i) \\ &+\varepsilon\left(\frac{1}{6\varepsilon} - \alpha\right)\Delta x^2\phi^{(3)}(x_i) + O(\Delta x^4) \end{aligned} \quad (7a)$$

$$\begin{aligned} &\phi''_i - \phi''(x_i) = (\varepsilon - 1)\phi''(x_i) \\ &+\varepsilon\left(\frac{1}{12} - \varepsilon\alpha\right)\Delta x^2\phi^{(4)}(x_i) + O(\Delta x^4) \end{aligned} \quad (7b)$$

where $\varepsilon = 1/(2\alpha + 2\beta)$. The accuracy is better when $\alpha + \beta = 1/2$. When $\alpha \neq 1/6$ and $\alpha \neq 1/12$, the first and second derivatives of approximate function have the accuracy of $O(\Delta x)^2$. A famous numerical method obtained if $\{\alpha, \beta\} \rightarrow \{0, 1/2\}$ is the FDM. When $\alpha = 1/6$ or $\alpha = 1/12$, the first and second derivatives of the approximate function have the accuracy of $O(\Delta x)^4$ and $O(\Delta x)^2$ or $O(\Delta x^2)$ and $O(\Delta x^4)$, respectively. The discrete relationship is defined as:

$$\begin{aligned} &\phi_i = \frac{p_{i-1} + 4p_i + p_{i+1}}{6}; \phi'_i = \frac{pi+1 - p_{i-1}}{2\Delta x}; \\ &\phi''_i = \frac{p_{i-1} - 2p_i + p_{i+1}}{\Delta x^2} + \Delta\phi''_i \end{aligned} \quad (8)$$

where $\Delta\phi'' = \frac{1}{12}\Delta x^2 \phi_i^{(4)} = \frac{p''_{i-1} - 2p''_i + p''_{i+1}}{12}$

Eq. (8) is the concept of the hybrid spline difference. It is simple to obtain Eq. (8) from Eq. (6), when using parameters $\{\alpha, \beta\} \to \{1/6, 1/3\}$; thus the accuracies of the first and second order derivatives can be increased to $O(\Delta x^4)$.

2.2. Mathematical Model and Numerical Procedure for Nonlinear Equation in Welding

2.2.1. Mathematical model

In order to validate whether the HSDM is applicable to determine temperature in ultrasonic welding process, the 2D welding model is considered in this study. According to the conservation of heat energy and Fourier's Law, the partial differential heat transfer equation

$$k(T)\frac{\partial^2 T}{\partial x^2} + k(T)\frac{\partial^2 T}{\partial y^2} + \frac{\partial k(T)}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 + \frac{\partial k(T)}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2 + q_w(x,y) + \rho C_p(T) V_w \frac{\partial T}{\partial x} = 0 \quad (9)$$

and accompanied boundary conditions

$$T(x,y) = T_\infty \; at \; x = x_{\max} and \; x = x_{\min} \quad (10a)$$

$$-k(T)\frac{\partial T(x,y)}{\partial y} = h(T - T_\infty) \; at \; y = y_{\max} \; and \; \frac{\partial T(x,y)}{\partial y} = 0 \; at \; y = y_{\min} \quad (10b)$$

where T and T_∞ are the temperatures in the specimen and surrounding temperature, respectively; ρ is the density, $k(T)$ and $C_p(T)$ are the thermal conductivity and heat capacity and x_{max}, x_{min} and y_{max}, y_{min} describe boundary dimensions of the work pieces in the x and y direction, respectively.

The thermal conductivity and heat capacity are functions of temperature, thus, in this problem becomes nonlinear. In Eq. (9) $q_w(x,y)$ is the heat generated during the welding process. The heat generated at the weld interface in the welding process is due to plastic deformation and friction from two work piece faces. According to Koellhoffer et al. [1] deformational heat generation is distinctively considered small in comparison to frictional heat generation for typical process parameters in USMW. Therefore, the temperature increase due to deformation is negligible. The frictional heat is indicated in:

$$q_w(x,y) = \frac{\mu F_N 2\xi_0 f_w}{w l_c} \quad (11)$$

where ξ_0 is the sonotrode amplitude, f_w is the welding frequency, F_N is the normal force and μ is the coefficient of friction.

2.2.2. Discretization and procedure of solving

Eq. (9) and Eq. (10) are discretized into the following form by using discrete mode in Eq. (8). The Hybrid Alternating Direction Implicit (HSADI) is used for 2D problem, the above equation is solved through exchange calculation after forward differential dispersion and the procedure is divided into two steps.

Step 1: solve x direction.

$$k(T)\left(\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \frac{p''_{i-1,j} - p''_{i,j} + p''_{i+1,j}}{12}\right) + \frac{\partial k(T)}{\partial T}\frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} \times \left(\frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} + \rho C_p(T) V_w\right) = -k(T)\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} - \frac{\partial k(T)}{\partial T}\left(\frac{\partial T}{\partial y}\right)^2_{i,j} - q_w(x,y) \quad (12)$$

Step 2: solve y direction.

$$k(T)\left(\frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} + \frac{p''_{i,j-1} - p''_{i,j} + p''_{i,j+1}}{12}\right) + \frac{\partial k(T)}{\partial T}\left(\frac{p_{i,+1j} - p_{i,j-1}}{2\Delta y}\right)^2 = -k(T)\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} - \frac{\partial k(T)}{\partial T}\frac{\partial T}{\partial x}\left(\frac{\partial T}{\partial x} + \rho C_p(T) V_w\right)_{i,j} - q_w(x,y) \quad (13)$$

Eqs. (12) and (13), can be further rearranged into the following forms

$$TA_i p_{i-1,j} + TB_i p_{i,j} + TC_i p_{i+1,j} = TD_i \quad (14)$$

In Eq. (14) TA_i, TB_i, TC_i *and* TD_i, *where* $i = 1, 2$, are known values, the iteration can be performed to determine new $p_{i,j}$ by using the Thomas algorithm [20] after $p_{-1,j}$ and $p_{N_i+1,j}$ are removed at step 1, $p_{i,-1}$ and p_{i,N_j+1} are eliminated at step 2. Then, Eq. (8) can be used to directly obtain the calculation discrete function $T(x,y)$ and its first- and second-order derivatives.

3. Numerical Results and Discussion

3.1. Example 1

The spline method described is used to solve the following differential equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = -2\pi^2 sin(\pi x) sin(\pi y) + \pi cos(\pi x) sin(\pi y) + \pi sin(\pi x) cos(\pi y) \quad (15a)$$

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With following boundary conditions:

$$T(0,y) = T(1,y) = T(x,0) = T(x,1) = 0 \quad \text{with } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \tag{15b}$$

Eq. (16) has an exact solution of

$$T(x,y) = sin(\pi x)sin(\pi y) \tag{16}$$

Using the discretization and solving procedure described in above section, the calculation steps are as below. First, solving in x direction:

$$\left(\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2} + \Delta T_{xx}\right) + \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} = -\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} - \frac{\partial T}{\partial y} - 2\pi^2 sin(\pi x)sin(\pi y) + \pi cos(\pi x)sin(\pi y) + \pi sin(\pi x)cos(\pi y) \tag{17a}$$

Second, solving in y direction

$$\left(\frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2} + \Delta T_{yy}\right) + \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y} = -\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} - \frac{\partial T}{\partial x} - 2\pi^2 sin(\pi x)sin(\pi y) + \pi cos(\pi x)sin(\pi y) + \pi sin(\pi x)cos(\pi y) \tag{17b}$$

Table 1. *Error comparison of proposed method, FDM, and parametric spline method*

$N_i \times N_j$	Finite Difference $(\alpha,\beta) \rightarrow (0,1/2)$	Parametric spline $(\alpha,\beta) \rightarrow (1/12,5/12)$	The present hybrid Spline
5 × 5	0.008932	0.0006458	5.077E-05
10 × 10	0.002759	0.0002039	5.251E-06
20 × 20	0.0007638	5.718E-05	4.507E-07
40 × 40	0.0002008	1.508E-05	3.362E-08
80 × 80	5.148E-05	3.869E-06	2.316E-09
160 × 160	1.303E-05	9.796E-07	1.585E-10

Table 1 shows a comparison of numerical errors obtained for three methods i.e. the proposed method, the parametric spline and the FDM. It can be clearly seen from Table 1 that the results obtained by HSDM have far higher numerical precision than those obtained by other methods. In addition, the speed of decreasing error of the HSDM shown to decrease significantly faster with increasing the grids number than that of the FDM and parametric spline methods. Additionally, numerical error and computing time comparison of the HSDM and FDM are shown in Table 2.

Table 2. *A comparison of numerical error and computer time between the proposed method and the finite difference method*

Grids	Error		Computer time (s)	
$N_i \times N_j$	Finite difference	The present	Finite difference	The present
5 × 5	0.008932	5.077E-05	0.001	0.02
10 × 10	0.002759	5.251E-06	0.009	0.042
20 × 20	0.0007638	4.507E-07	0.122	0.382
40 × 40	0.0002008	3.362E-08	1.622	5.393
80 × 80	5.148E-05	2.316E-09	21.706	80.196
160 × 160	1.303E-05	1.585E-10	290.37	1211.5

As shown in in Table 2, the numerical error is 1.303E-05 when grid points of $N_i \times N_j = 160 \times 160$ was selected for solution using FDM. When the proposed method is used, the error can rea ch 5.077E-05 with grid points of $N_i \times N_j = 5 \times 5$ and the calculation time can be reduced from 290.37 to 0.02 s. The results indicate that the presented method is superior to both FDM and parametric spline method and can rapidly reduce the computer time.

3.2. Example 2

Example 2 considers solving the nonlinear equation of ultrasonic welding problem as described by Eqs. (9), (10a) & (10b). The welding conditions are set as sonotrode amplitude, $\xi_0 = 32 \times 10^{-6} m$, welding frequency, $f_w = 20000$, and normal force, $F_N = 1600N$.

Because the governing equation is nonlinear, the exact analytic solution is simulated by 20 times the grid point numbers in the case of an unavailable analytic solution.

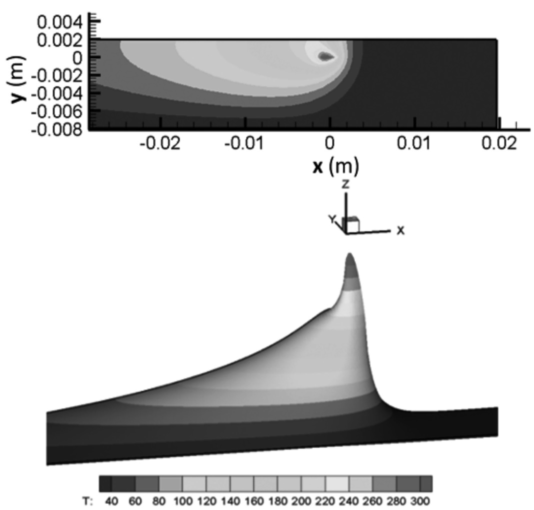

Fig.1. *Distribution of the calculated temperature*

Figure 1 shows temperature distribution in the work pieces with a size of *50 mm × 10 mm × 1mm* resulting from a moving heat source with respect to direction at 35.5mm/s travelling speed.

It is observed that the temperature is significantly concentrated on the welding zone and the highest temperature is seen at the interface.

Figures 2 and 3 show the numerical solutions obtained by different methods with grid number $N_i \times N_j = 60 \times 30$ and $N_i \times N_j = 300 \times 150$, respectively.

It is seen that the error amplitude of numerical solution of all methods decrease with increasing the number of grid points.

Since this problem does not have an analytic solution, the error is calculated based on the results of $N_i \times N_j = 1200 \times 600$ grids which called exact solution.

As shown in Figures 2 and 3, the solution of HSDM is the closest to exact result, then the parametric spline method and FDM.

Table 3 shows that the accuracy of the HSDM method is higher than that of the FDM. - The result of the proposed method is more accurate than that of the parametric spline result.

According to Figures 2-3 and Table 3, the FDM has the worst numerical precision, but the result of parametric spline method is significantly better than that when the parameter $(\alpha, \beta) \to (1/12, 5/12)$.

However, the result obtained through the HSDM is not only better, but also clearly faster decreased error speed than that obtained from any method presented in this article.

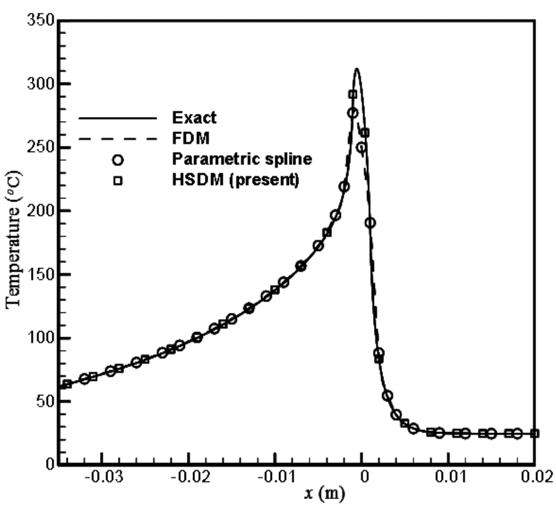

Fig.2. *Numerical solution of different numerical method (grid number* $N_i \times N_j = 60 \times 30$*)*

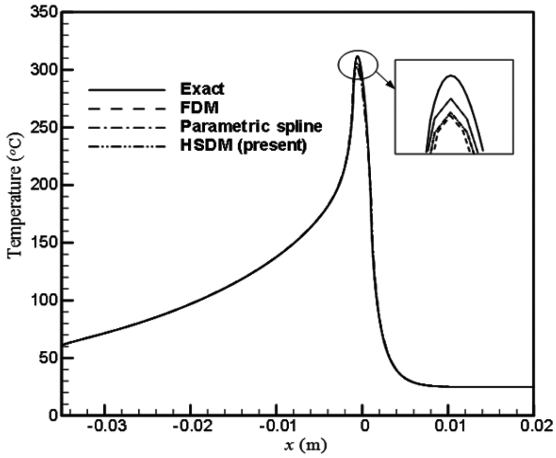

Fig.3. *Numerical solution of different numerical method (grid number* $N_i \times N_j = 300 \times 150$*)*

Table 3. *Error comparison of proposed method, FDM, and parametric spline method*

$N_i \times N_j$	Finite Difference $(\alpha, \beta) \to (0, 1/2)$	Parametric spline $(\alpha, \beta) \to (1/12, 5/12)$	The present hybrid Spline
60 × 30	1.4756	1.4760	1.2590
120 × 60	0.9493	0.9491	0.9033
240 × 120	0.5237	0.4256	0.06292
300 × 150	0.3951	0.0993	0.01795
600 × 300	0.1022	0.0579	0.000452

It can be seen that the numerical error of all methods of example 2 is greater than that of example 1. Because the example 2 is nonlinear and the heat generation with discontinuous parameter, only distributes at a defined range. Although, this makes example 2 complex and its numerical error rising, with increasing grids number, the proposed method still has high accuracy and achieves the good results.

4. Conclusions

The proposed method could successfully solve a nonlinear equation of welding problem in ultrasonic welding and proved that the HSDM can simplify complicate calculation procedure of traditional spline theory.

Interestingly, its discretization instruction is very similar to the FDM; however, its accuracy is significantly enhanced. The temperature distribution in the work pieces, found by applying current method, well agrees with the "exact analytic" solution. Accordingly, not only the HSDM, proposed in this article, is a simple and potential numerical method for solving non-linear differential equations, but also could be a potential candidate for replacement of the traditional FDM.

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GIẢI BÀI TOÁN TRUYỀN NHIỆT TRONG HÀN SIÊU ÂM DỰA TRÊN PHƯƠNG PHÁP SAI PHÂN KẾT HỢP ĐƯỜNG CONG

Tóm tắt:
Phương pháp sai phân kết hợp đường cong được phát triển để giải bài toán phi tuyến trong hàn siêu âm. Phương pháp này có quy trình tính toán đơn giản như phương pháp sai phân hữu hạn. Ngoài ra, phương pháp được đề xuất có thể đơn giản hóa độ phức tạp của các biểu thức của phương pháp đường cong truyền thống và tăng độ chính xác của các đạo hàm bậc nhất và bậc hai theo không gian từ $O(\Delta x^2)$ *của phương pháp sai phân hữu hạn sang* $O(\Delta x^4)$*. Theo sự phân bố nhiệt độ tính toán trong chi tiết hàn trong quá trình hàn siêu âm, phương pháp đề xuất đã cho thấy rằng không chỉ độ chính xác của phương pháp được nâng cao đáng kể, mà khái niệm của nó cũng rất giống với phương pháp sai phân hữu hạn. Dựa trên kết quả phân tích, ta đã kết luận rằng phương pháp sai phân kết hợp đường cong là đơn giản và có độ chính xác cao, có tiềm năng tốt để thay thế phương pháp sai phân hữu hạn truyền thống.*

Từ khóa: *Hàn siêu âm kim loại, phương pháp sai phân kết hợp đường cong, phương pháp sai phân hữu hạn.*