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ON THE ASYMPTOTICS OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract:
The asymptotic form of solution is a specific solution of difference equations. It represents the rate of convergence of solution to equilibrium point of difference equations. In this paper, we investigate asymptotic form of a class of nonlinear difference equations. By asymptotic method, we show the existence of a solution of a class of difference equations converging to zero as $n \to \infty$*, and to determine its asymptotic behavior.*
Keywords: *Difference equations, Asymptotics behavior.*

1. Introduction

In recently year, althought many authors interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations, some authors study the existence of specific solutions of difference equations, such as monotonous, non-equilibrium, etc. While solving these problems, we developed various asymptotic methods, see [1-12].

In [2], L. Berg shown the existence of a solution converging to zero of the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{1+x_n}.$$

The equation above can be written in the following form

$$x_{n-1} = x_{n+1}(1+x_n).$$

In [5], L. Berg proved the existence of a solution, and to determine its asymptotic behavior the difference equation:

$$x_{n-3} = x_n(1+x_{n-1}x_{n-2}).$$

In [6], L. Berg and S. Stevi’c show that the difference equation

$$y_{n-k} = y_n(1+y_{n-1}\dots y_{n-k+1}).$$

where $k \in N \backslash \{1\}$, has a positive solution converging to zero, by finding a finite asymptotic expansion of the solution.

Motivated by the aforementioned study, our goal in this paper is to investigate the asymptotic behavior of solutions of some classes of difference equations

$$x_{n-1} = x_{n+1} f(x_n), \quad (1.1)$$

where f is the polynomial of x_n of the form

$$f(x_n) = 1 + a_1 x_n + a_2 x_n^2.$$

with the parameters a_i for $i \in \{1,2\}$ are arbitrary real numbers.

2. Preliminaries

In order to prove the main results of the paper, we use the ideas from [5]. For this, we recall the following inclusion theorem from [11], which is a natural extension of Theorem 1 in [2].

Theorem 2.1. *Let* $f : R^{k+1} \to R$ *be a continuous and nondecreasing function in each argument, and let* y_n *and* z_n *be sequences such that* $y_n < z_n$ *for* $n \ge n_0$ *and*

$$y_{n-k} \le f(y_{n-k+1},\dots,y_{n+1}),\ f(z_{n-k+1},\dots,z_{n+1}) \le z_{n-k} \quad (2.1)$$

for $n \ge n_0 + k - 1$. *Then, the difference equation*

$$x_{n-k} = f(x_{n-k+1},\dots,x_{n+1}) \quad (2.2)$$

has a solution x_n *such that*

$$y_n \le x_n \le z_n \text{ for } n \ge n_0.$$

Theorem 2.1 can be applied as follows. Let the functions $\phi_1, \phi_2, \dots,$ form an asymptotic scale as $n \to \infty$, i.e.

$$\phi_{i+1} = o(\phi_i),$$

(cf. [1]), and look for a finite asymptotic expansion

$$x_n(a) = \phi_0 + a_1\phi_1 + \dots + a_{N-1}\phi_{N-1} + a\phi_N + o(\phi_N),$$

with a fixed N.

Let

$$F(x_{n-k},...,x_{n+1}) = f(x_{n-k+1},...,x_{n+1}) - x_{n+1},$$

and

$$F(x_{n-k},\ldots,x_{n+1}) \sim (a_N - a)\emptyset_N,$$

with $\emptyset_N > 0$.

Then, we can apply Theorem 2.1 with

$y_n = x_n(a)$ for an $a < a_N$,

$z_n = x_n(a)$ for an $a > a_N$,

and $x_n = x_n(a_N)$ is a finite asymtotic expansion of solution x_n of difference equation (2.2).

3. Main results

In this section, we state and prove the main result of this paper.

Theorem 3.1. *If $a_1 \neq 0$, then equation (1.1) has a solution with the following asymptotics*

$$x_n = \frac{1}{n}\left[a + \frac{b\ln n + c}{n} + \frac{d\ln^2 n}{n^2} + o\left(\frac{\ln^2 n}{n^2}\right)\right], \quad (3.1)$$

where

$$a = \frac{2}{a_1}, b = \frac{2}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right),$$

$$c = \frac{1}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right), d = \frac{1}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right)^2.$$

Proof. Note that equation (1.1) can be written in the following equivalent form:

$$F(x_{n-1},x_n,x_{n+1}) = x_{n+1}(1 + a_1 x_n + a_2 x_n^2) - x_{n-1} = 0. \quad (3.2)$$

First, according to the approach in [5], we assume that the solutions of (1.1) have finite asymtotic expansion of the form:

$$\phi_n = \frac{1}{n}\left[a + \frac{b\ln n + c}{n} + \frac{d\ln^2 n}{n^2}\right], \quad (3.3)$$

Then by mean DERIVE, we find that

$$F(\phi_{n-1},\phi_n,\phi_{n+1}) = \phi_{n+1}(1 + a_1\phi_n + a_2\phi_n^2) - \phi_{n-1}$$
$$= \left[\phi_n + \phi'_n + o\left(\frac{\ln n}{n^4}\right)\right](1 + a_1\phi_n + a_2\phi_n^2)$$
$$-\left[\phi_n - \phi'_n + o\left(\frac{\ln n}{n^4}\right)\right]$$
$$= (\phi_n + \phi'_n)(a_1\phi_n + a_2\phi_n^2) + 2\phi' + o\left(\frac{\ln n}{n^4}\right).$$

From (3.3), we have

$$\phi_n^2 = \frac{a^2}{n^2} + \frac{2ab\ln n + 2ac}{n^3} + \frac{(b^2 + 2ad)\ln^2 n + 2bc\ln n}{n^4} + o\left(\frac{\ln n}{n^4}\right).$$

$$\phi'_n = -\frac{a}{n^2} + \frac{(-2b)\ln n + b - 2c}{n^3} + \frac{(-3d)\ln^2 n + 2d\ln n}{n^4} + o\left(\frac{\ln n}{n^4}\right).$$

From that, we have

$$F(\phi_{n-1},\phi_n,\phi_{n+1}) = \frac{a_1 a^2 - 2a}{n^2} + \frac{(2a_1 ab - 4b)\ln n}{n^3}$$
$$+ \frac{a_2 a^3 + 2a_1 ac - a_1 a^2 + 2b - 4c}{n^3} +$$
$$+ \frac{(a_1 b + 2a_1 ad - 6d)\ln^2 n}{n^4} +$$
$$+ \frac{(3a_2 a^2 b - 3a_1 ab + 2a_1 bc + 4d)\ln n}{n^4} + o\left(\frac{\ln n}{n^4}\right). \quad (3.4)$$

Next, with condition $a_1 \neq 0$, by equating the coefficients in (3.4) to zero, we have

$$a = \frac{2}{a_1}, b = \frac{2}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right),$$
$$c = \frac{1}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right), d = \frac{1}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right)^2. \quad (3.5)$$

Using the value of *a, b, c,* we get

$$F(\phi_{n-1},\phi_n,\phi_{n+1}) = 4\left[d - \frac{1}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right)^2\right]\frac{\ln n}{n^4} + + o\left(\frac{\ln n}{n^4}\right),$$

or

$$F(\phi_{n-1},\phi_n,\phi_{n+1}) = h(d)\frac{\ln n}{n^4} + o\left(\frac{\ln n}{n^4}\right),$$

where

$$h(t) = 4\left[t - \frac{1}{a_1}\left(1 - \frac{2a_2}{a_1^2}\right)^2\right].$$

Since $h(d) = 0$, then we can choose d_1, d_2 so that

$$h(d_1) > 0, h(d_2) < 0.$$

While, we choose

$$y_n = \frac{a}{n} + \frac{b\ln n + c}{n^2} + \frac{d_1\ln^2 n}{n^3},$$
$$z_n = \frac{a}{n} + \frac{b\ln n + c}{n^2} + \frac{d_2\ln^2 n}{n^3},$$

we get

$$F(y_{n-1}, y_n, y_{n+1}) \sim h(d_1)\frac{\ln n}{n^4} > 0$$

and

$$F(z_{n-1}, z_n, z_{n+1}) \sim h(d_2)\frac{\ln n}{n^4} < 0,$$

for sufficiently large n. According Theorem 2.1, it follows that there exists a solution of equation (3.2) having the finite asymtotic expansion (3.3) with the coefficients in (3.5). So the proof is complete. □

Remark 3.1. For $a_1 = 1$, $a_2 = 0$, the result of Theorem 3.1 is well known from [2].

4. Conclusion

This work is related to asymptotic behavior of a class of nonlinear difference equations. We have proved that equation (1.1) possesses a solution with the finite asymptotic expansion (3.3) and the coefficients (3.5). Moreover, that asymptotic form converging to zero as $n \to \infty$.

Open problem: We can find the asymptotic behavior of solutions of a class of difference equations

$$x_{n-1} = x_{n+1} f(x_n),$$

where f is the polynomial of x_n of the form

$$f(x_n) = 1 + a_1 x_n + a_2 x_n^2 + ... + a_m x_n^m.$$

with the parameters a_i for $i \in \{1,2,\ldots,m\}, m \in \mathbf{N}^*$ are arbitrary real numbers.

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DẠNG TIỆM CẬN CỦA MỘT LỚP PHƯƠNG TRÌNH SAI PHÂN PHI TUYẾN

Tóm tắt:

Dạng tiệm cận của nghiệm là một nghiệm cụ thể của phương trình sai phân. Nó biểu diễn tốc độ hội tụ của nghiệm đến điểm cân bằng của phương trình sai phân. Trong bài báo này, chúng tôi nghiên cứu dạng tiệm cận của một lớp phương trình sai phân phi tuyến. Sử dụng phương pháp tiệm cận, chúng tôi chứng tỏ sự tồn tại nghiệm của một lớp phương trình sai phân hội tụ về không khi $n \to \infty$ *và xác định dáng điệu tiệm cận của nó.*

Từ khóa: *Phương trình sai phân, Dáng điệu tiệm cận.*